Polarized Preferences in Homegrown Value Auctions

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Homegrown value-auction experiments are useful for exploring preferences for controversial product attributes. These auctions have emphasized estimating the effect of the attribute on the willingness to pay (WTP) for a product. The likelihood that individuals are willing to purchase any products with the attribute has received less attention, even though this could also be useful to researchers, marketers, and policy makers. This article shows how simultaneous, single-unit auctions can be used to estimate not just WTPs, but also the likelihood that individuals are willing to purchase any products with a controversial attribute.

Key words: consumer preferences, experimental economics, homegrown value auctions, invasive species, polarization, willingness to pay

Introduction

Polarization is common in politics. Some individuals refuse to vote for any Republican candidate (negative polarization), while others will only vote for Republican candidates (positive polarization). But polarization is not confined to politics. Recent decades have seen a parade of controversial product attributes that have also been polarizing. Some consumers refuse to buy any products made with genetically engineered crops, and others will only buy produce that is certified organic. While some attributes like organic certification have enjoyed commercial success, others like irradiation have struggled. The past few decades have also seen the introduction of product labels in an effort to discourage consumption of products with socially undesirable attributes or encourage consumption of products with socially desirable attributes. Examples include graphic warning labels on tobacco products and the American Heart Association’s Heart Check logo on products meeting specific nutritional guidelines. Therefore, researchers, marketers, and policy makers could benefit from having effective strategies for gauging the extent of polarization to an attribute that is shared by many products.

Homegrown value-auction experiments are useful for exploring consumer preferences for controversial product attributes (e.g., Buhr et al., 1993; Fox et al., 1994; Shogren et al., 1994; Fox et al., 1998; Roosen et al., 1998; Lusk et al., 2001; Fox, Hayes, and Shogren, 2002; Alfnes and Rickertsen, 2003; Huffman et al., 2003; Lusk et al., 2004; Ward, Bailey, and Jensen, 2005; Wachenheim, Lambert, and Van Wechel, 2007; Rousu et al., 2007; Thrasher et al., 2007). These experiments use incentive-compatible auctions to elicit the maximum willingness to pay (WTP) for a product or to trade one product for another.¹ Users of these auctions have been aware of polarization

¹ Alternatively, some researchers have elicited minimum willingness to accept (WTA).
as a result of the controversial nature of the attributes of interest and have devised a variety of strategies to control for it using their experimental design or analysis. For example, Buhr et al. (1993) used a split-sample design that endowed some individuals with a typical meat sandwich that could be traded for a leaner, growth-hormone-treated meat sandwich, while others were endowed with a leaner meat sandwich that could be traded for the typical meat sandwich. Fox et al. (1998) took this design a step further by pretesting to determine who would be endowed with a typical and who would be endowed with an irradiated pork sandwich. Parkhurst, Shogren, and Dickinson (2004) explored the potential for using a design that permitted negative bids. To econometrically account for the zero bids that are symptomatic of negative polarization, Roosen et al. (1998) and Lusk et al. (2001) used a double-hurdle econometric model; Fox, Hayes, and Shogren (2002) used a Tobit; Corrigan and Rousu (2006) used a random effects Tobit; and Lusk (2010) used a simultaneous equations, seemingly unrelated (SUR) Tobit. We are unaware of other attempts to use experimental auctions to specifically quantify polarization.

This article illustrates how experimental auctions can be used to quantify polarization in addition to estimating WTP. To accomplish this objective, we formally characterize polarization and bidding behavior in an experimental auction. We then use our characterization to show the effect that auction designs with one product traded for another have on bidding behavior and how a finite-mixture econometric model can be used to quantify polarization with simultaneous, single-unit auction designs. Finally, we test the proposed finite-mixture model using data from an ornamental plant auction that was designed to explore the potential for using informational labels to reduce invasive-plant purchases. The contributions include a new framework for characterizing bidding behavior in homegrown value auctions; a demonstration of the feasibility of using simultaneous, single-unit auctions with a finite-mixture model to quantify polarization; and the quantification of polarization to invasive and noninvasive ornamental plants with and without informational labels.

**Preferences and Polarization**

One way to think about the polarization of a controversial attribute that is shared by many products is in the context of “attribute nonattendance” or some other form of lexicographic preference in which the attribute is the first thing considered by a consumer, who ignores other attributes (Scarpa et al., 2009). However, lexicographic preferences are the textbook example of preferences that cannot be characterized by a real-valued utility function, which means the neoclassical results from consumer theory do not apply. Fortunately, the assumption of lexicographic preferences is stronger than necessary to characterize polarization. All that is really needed is local nonsatiation, which permits undesirable products provided there are also some desirable products.

Let \( x \in R_+^K \) for \( K > 1 \) be a vector of products that share a particular attribute—like being made from genetically engineered crop ingredients—and \( y \in R_+^L \) for \( L > 0 \) be a vector of all other products without this attribute. Let \( z = (x, y) \) be the vector of all products. Assume preferences are rational, locally nonsatiated, and continuous so that they can be represented by a continuous real-valued utility function \( U(z) \). Let \( z_{-h} \) be the vector \( z \) exclusive of \( z_h \).

**Definition 1:** A product \( z_h \) is strictly desirable (undesirable) if for all \( z, z' \in R_+^{K+L} \), \( z'_h > z_h \) and \( z'_{-h} = z_{-h} \) imply that \( U(z') > (\leq) U(z) \).

An individual always prefers to consume more of a strictly desirable product and less of a strictly undesirable one.

**Definition 2:** The attribute defining the vector \( x \) is positively (negatively) polarized if \( x_k \) is strictly desirable (undesirable) for all \( k = 1, \ldots, K \). The attribute is not polarized if it is not positively polarized and it is not negatively polarized.

An individual’s preferences are positively (negatively) polarized for an attribute if all the products with the attribute are strictly desirable (undesirable). An individual’s preferences are not polarized for an attribute if some products with the attribute are desirable but others are undesirable depending on the consumption of other products.
Behavior in Experimental Auctions

Behavior in an experimental auction can be characterized with modest embellishments to neoclassical consumer theory recognizing that opportunities “in the field” (i.e., outside the experiment) can influence behavior within the experiment (Harrison, Harstad, and Rutstrom, 2004). Let \( m^0 \in \mathbb{R}_{++}^n \) be income and \( p \in \mathbb{R}_{+}^{K+L} \) be prices in the field. Assume an individual is endowed with or is awarded \( z_h^e \in \mathbb{R}_+^n \) and \( m^e \in \mathbb{R}_+ \) within the experiment, such that \( z_h^e \) cannot be resold “in the field” but \( m^e \) can be spent “in the field.” Note that \( m^e \) is also assumed to be available within the experiment.

Budget Sets and Indirect Utility

Consumption opportunities depend on whether \( z_h^e \) is freely disposable. In some auction experiments, individuals are not required to consume products and can dispose of them after leaving. In others, consumption is required before leaving. Let \( p_{-h} \) be the price vector exclusive of the price of \( z_h \) and \( m = m^0 + m^e \). The budget set with free disposal is

\[
B_h(z_h^e, p, m) = \{ z \in \mathbb{R}_+^{K+L} : p \cdot z \leq m + p_hz_h^e \text{ and } p_{-h} \cdot z_{-h} \leq m \}.
\]

The first constraint in equation (1) captures the notion that \( z_h^e \) makes it possible for the individual to consume more because some income that might have been spent on \( z_h \) can be reallocated to other goods. However, the benefits of this extra consumption are limited if \( z_h^e \) has no resale value and the individual is not interested in consuming all of \( z_h^e \)—the second constraint. The budget set without free disposal is

\[
B_h(z_h^e, p, m) = \{ z \in \mathbb{R}_+^{K+L} : p \cdot z \leq m + p_hz_h^e \text{ and } z_h \geq z_h^e \}.
\]

The second constraint now implies that the individual must consume at least \( z_h^e \) of \( z_h \).

The optimal consumption set assuming utility maximization is

\[
z_h(z_h^e, p, m) = \{ z \in B_h(z_h^e, p, m) : U(z) \geq U(z') \text{ for all } z' \in B_h(z_h^e, p, m) \}.
\]

Since \( B_h(z_h^e, p, m) \) is a compact, convex set and \( U(z) \) is a continuous real-valued function, this consumption set is nonempty. Given this optimal consumption set, the indirect utility function is

\[
V_h(z_h^e, p, m) = U(z) \text{ for any } z \in z_h(z_h^e, p, m).
\]

The indirect utility function in equation (4) differs from the typical neoclassical indirect utility function because it depends on \( z_h^e \) as well as prices and income. How this indirect utility function depends on income and \( z_h^e \) is important for determining the optimal bid in an incentive-compatible auction. Like the typical neoclassical indirect utility, this indirect utility is increasing in income. If \( z_h \) is strictly desirable, increasing \( z_h^e \) increases indirect utility because it is as if the individual had more income. If \( z_h \) is strictly undesirable, increasing \( z_h^e \) has no effect on the indirect utility with free disposal because an individual can simply dispose of \( z_h^e \). Without free disposal, the indirect utility is decreasing in \( z_h^e \) because individuals must consume the undesirable product.

Optimal Bids and WTP

The optimal bid in an incentive-compatible auction can be determined by first finding the WTP. If the endowed product \( z_h^e \geq 0 \) can be traded for the auctioned product \( z_h^e > 0 \), the WTP is

\[
d^h(z_h^e, p, m) = \{ z \in B_h(z_h^e, p, m) : p \cdot z = m + p_hz_h^e \text{ and } p_{-h} \cdot z_{-h} \geq m + p_hz_h^e \} \; \text{ and } p_{-h} \cdot z_{-h} = m + p_hz_h^e \text{ instead}.
\]

The properties of the indirect utility function with an endowment in terms of changes in income and the endowment are formally stated in the appendix as Propositions A1 and A2.
which is the most income that can be given up to trade \( z^*_h \) for \( z^*_f \) without reducing utility. Since indirect utility is increasing in income, the WTP can also be written as \( w(z^*_h, z^*_f) = m - e_h(z^*_h, p, V(z^*_f, p, m)) \) where \( e_h(z^*_h, p, U^o) = V^{-1}_h(z^*_h, p, U^o) \) is the expenditure function given \( z^*_h \) prices and some arbitrary utility \( U^o \). The vast majority of experimental auctions require nonnegative bids, implying that the optimal bid in an incentive-compatible auction is

\[
b(z^*_h, z^*_f) = \max\{w(z^*_h, z^*_f), 0\}.
\]

With a nonnegativity requirement on bidding behavior and free disposal, equations (5) and (6) imply that the optimal bid equals the WTP, so the auction is in fact incentive compatible. Without free disposal, the optimal bid need not equal the WTP, so the auction need not be incentive compatible.

For example, suppose \( z^*_h \) is a typical meat sandwich and \( z^*_f \) is a leaner meat sandwich as a result of growth hormones. Without free disposal, if \( z_h \) is strictly desirable and \( z_l \) is strictly undesirable, equations (5) and (6) imply that \( b(z^*_h, 0) = w(z^*_h, 0) > 0 \) and \( b(z^*_f, 0) = 0 > w(z^*_f, 0) \). The optimal bid for the desirable product is an unbiased estimate of the WTP, while the optimal bid for the undesirable product is upward biased. Alternatively, if an individual is endowed with \( z^*_f \) and given the opportunity to trade for \( z^*_h \), then equations (5) and (6) imply that \( b(z^*_h, z^*_f) = w(z^*_h, z^*_f) \), such that the optimal bid is an unbiased estimate of the WTP to trade the endowed product for the auction product. Therefore, endowing individuals with a strictly undesirable product and letting them trade for a strictly desirable product may provide a way to obtain an unbiased estimate of the difference in the WTP between the auctioned and endowed products, even though the bid for the strictly undesirable product will be biased when auctioned separately. However, for this to be true, \( w(z^*_h, 0) - w(z^*_f, 0) = w(z^*_h, z^*_f) \).

The hypothesis that \( w(z^*_h, 0) - w(z^*_f, 0) = w(z^*_h, z^*_f) \) can be explored analytically using our framework. Equation (5) implies

\[
w(z^*_h, 0) - w(z^*_f, 0) / = / w(z^*_h, z^*_f) \text{ when (7) } e_l(z^*_f, p, V(p, m)) - e_h(z^*_h, p, V(p, m)) > / = / e_l(z^*_f, p, V(z^*_f, p, m)) - e_h(z^*_h, p, V(z^*_f, p, m)),
\]

where \( V(p, m) \) is the indirect utility without an endowment. Equation (7) shows that the hypothesis can be evaluated in terms of neoclassical substitution effects (i.e., how much income would have to change to consume more of one product and less of another holding utility constant) with and without an endowment.

The left side of figure 1 illustrates a case where \( w(z^*_h, z^*_f) > w(z^*_h, 0) - w(z^*_f, 0) \) because \( z_h \) becomes relatively more attractive with an endowment of \( z^*_f \), while the right side illustrates a case where \( w(z^*_h, 0) - w(z^*_f, 0) > w(z^*_h, z^*_f) \) because \( z_h \) becomes relatively less attractive with an endowment of \( z^*_f \). Both illustrations assume free disposal, but are equally applicable to cases without free disposal with some minor modifications to the illustrated budget sets. At point f, the indifference curve corresponding to utility \( V(p, m) \) is tangent to the budget set corresponding to the expenditure without an endowment \( e(p, V(p, m)) = m \), which reflects the optimal consumption of \( z_h \) and \( z_l \) in the absence of an auction and any product endowment. Point a shows the consumption required to achieve utility \( V(p, m) \) given that \( z^*_f \) is awarded in the auction. The minimum expenditure needed to reach this point given \( z^*_f \) is reflected by the budget set corresponding to \( e_l(z^*_f, p, V(p, m)) \) implying that \( w(z^*_f, 0) = p_l(z^*_f - z^*_f) = m - p_h z^*_h \). Similarly, point b shows the consumption required to achieve utility \( V(p, m) \) given that \( z^*_h \) is awarded in the auction, implying a minimum expenditure of \( e_h(z^*_h, p, V(p, m)) \) and \( w(z^*_h, 0) = m - p_l(z^*_h - z^*_h) = m - p_h(z^*_h + z^*_h - z^*_h) \). The resulting difference in these WTPs is \( w(z^*_h, 0) - w(z^*_f, 0) = p_l(z^*_f - z^*_h - z^*_f) = p_h(z^*_h + z^*_h - z^*_h) \).

With an endowment of \( z^*_f \) utility \( V_l(z^*_f, p, m) \) can be achieved in the absence of the auction, implying consumption at point c with expenditure \( e_l(z^*_f, p, V(z^*_f, p, m)) = m \). Given the opportunity to trade

\footnote{Having \( z_h \) be strictly desirable and \( z_l \) be strictly undesirable is sufficient but not necessary. Necessity only requires \( V_h(z^*_h, p, m) > V_h(0, p, m) \) and \( V_l(z^*_f, p, m) < V_l(0, p, m) \).}
Second-price and random price auctions. Furthermore, they found that second-price and random mechanisms and products. Lusk, Feldkamp, and Schroeder (2004) rejected the hypothesis for Harstad, and Rutstrom, 2004). Alternatively, if the endowed product is strictly undesirable, the estimate of the difference in WTPs between two products depends on the difference in expenditures between points a and b compared to the difference in expenditures between points c and d.

Figure 1 illustrates why auctions with endowments will not generally provide an unbiased estimate of the difference in WTPs between two products and how the direction of any bias will depend on individual preferences. However, there are special cases where \( w(z^e_h,0) = w(z^e_f,0) \) will hold for sure. For example, if it is optimal for individuals to consume \( z_h > z^e_h \) and \( z_l > z^e_l \) given prices and income, equation (5) implies that \( w(z^e_h,0) = p_h z^e_h \), \( w(z^e_f,0) = p_f z^e_f \), and \( w(z^e_h, z^e_f) = p_h z^e_h - p_l z^e_l \) because \( V_l(z^e_h, p, m) = V(p, m + p_l z^e_l) \). Hence, \( e(z^e_h, z^e_f) = e(p, U^o) - p_h z^e_h \), and \( e_l(z^e_f, p, U^o) = e(p, U^o) - p_l z^e_f \). While endowment auctions will not yield biased estimates of the difference in WTPs in such instances, these WTPs will be field-price censored instead (Harrison, Harstad, and Rutstrom, 2004). Alternatively, if the endowed product is strictly undesirable, then \( V_l(z^e_h, p, m) = V(p, m) \) such that equation (7) holds with equality. Therefore, auctions with endowments—such as those described in Buhr et al. (1993) and Fox et al. (1998)—can yield unbiased estimates of the difference in WTPs between a strictly desirable auctioned product and strictly undesirable endowed product, but not necessarily between two strictly desirable products.

Lusk, Feldkamp, and Schroeder (2004) and Corrigan and Rousu (2006) explore the hypothesis \( w(z^e_h,0) = w(z^e_f,0) = w(z^e_h, z^e_f) \) experimentally rather than analytically using a variety of auction mechanisms and products. Lusk, Feldkamp, and Schroeder (2004) rejected the hypothesis for second-price and random nth-price auctions, but not for English or Becker-DeGroot-Marshak random price auctions. Furthermore, they found that \( w(z^e_h, z^e_f) > w(z^e_h,0) = w(z^e_f,0) \) for the second-price auction and \( w(z^e_h,0) - w(z^e_f,0) > w(z^e_h, z^e_f) \) for the random nth-price auction. Corrigan and Rousu (2006) also rejected this hypothesis using second-price and random nth-price auctions, though they consistently found \( w(z^e_h, z^e_f) > w(z^e_h,0) = w(z^e_f,0) \). Both looked beyond neoclassical theory to
explain their results. While Lusk, Feldkamp, and Schroeder (2004) did not consider a neoclassical explanation, Corrigan and Rousu (2006) argued that their small and insignificant estimate for the effect of income on the WTP could not explain the observed differences in bidding behavior. However, equation (7) indicates that such an income effect is irrelevant in terms of the hypothesis of interest, meaning their dismissal of a neoclassical explanation may have been premature.

Identifying Polarized Preferences

Our characterization of polarization and behavior in experimental auctions allows us to explore the potential for using these experiments to identify the probability that an attribute is positively or negatively polarized. One of the implications of this framework is that the only distinguishing characteristic of bids in terms of polarization is whether they are positive or zero. Based on this implication, we first consider the information about polarization conveyed by bids from a single-unit auction. We then consider the information conveyed by bids from simultaneous, single-unit auctions with products that share a polarizing attribute.

Single-Unit Auctions

Single-unit auctions are rarely conducted in isolation. They are typically repeated for a number of rounds or conducted simultaneously with other single-unit auctions. Still, even with these repeated or simultaneous, single-unit auctions, the goal is to provide subjects with an incentive to treat each auction independently.

The information about polarization conveyed by bids from a single-unit auction depends on whether there is an endowment. With an endowment, the information conveyed also depends on the characteristics of the endowed and auctioned products and on free disposal. Table 1 summarizes the information about polarization conveyed by bids for a variety of auction designs based on the implications of optimal bidding behavior that follow from equations (5) and (6).

The first result to note from table 1 is that neither bid behavior completely identifies polarization, regardless of the auction design. Since individuals without polarization may choose a positive or zero bid depending on other products consumed, their bid behavior is confounded with what an individual with positive or negative polarization would do; that is, there is no signal that distinguishes an individual without polarization from one with positive or negative polarization. Second, auctions without product endowments are as informative as and often more informative than auctions with product endowments because they make it possible to discern whether an individual is not negatively polarized or not positively polarized. Auctions with an endowment are typically less informative because optimal bids for positive and negative polarization can also become confounded. Third, all but one of the auction designs provide at least some information on polarization. For an auction with free disposal where the auctioned product is strictly undesirable and the endowed product has the potentially polarized attribute, the optimal bid is zero regardless of polarization, so it is completely

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6 These implications are formally stated and proved in Propositions A3–A5 in the appendix.
Table 1. Information about Polarization Conveyed by Positive and Zero Bids in Different Single-Unit Auction Designs

<table>
<thead>
<tr>
<th>Single-Unit Auction Designs</th>
<th>Information Conveyed about Polarized Attribute From Positive Bid</th>
<th>Zero Bid</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free Disposal</td>
<td>Auctioned Product</td>
<td>Endowed Product</td>
</tr>
<tr>
<td>Yes/No Polarized Attribute</td>
<td>None</td>
<td>±</td>
</tr>
<tr>
<td>Yes/No Polarized Attribute</td>
<td>Strictly Desirable</td>
<td>±, ±</td>
</tr>
<tr>
<td>Yes/No Polarized Attribute</td>
<td>Polarized Attribute</td>
<td>±, ±</td>
</tr>
<tr>
<td>Yes Polarized Attribute</td>
<td>Strictly Undesirable</td>
<td>±</td>
</tr>
<tr>
<td>No Polarized Attribute</td>
<td>Strictly Undesirable</td>
<td>±</td>
</tr>
<tr>
<td>Yes Polarized Attribute</td>
<td>Polarized Attribute</td>
<td>{−, ±, +}</td>
</tr>
<tr>
<td>No Polarized Attribute</td>
<td>Polarized Attribute</td>
<td>{−, ±, +}</td>
</tr>
</tbody>
</table>

Notes: + indicates the attribute is positively polarized, − indicates the attribute is negatively polarized, and ± indicates the attribute is not polarized.

uninformative. Finally, it may be possible to further refine these information sets if some information is known about polarization a priori. For example, if the attribute is known not to be negatively polarized in an auction without an endowment, then a positive bid would still indicate {±, +}, but a zero bid would be completely informative for no polarization, indicating {±} instead of {−, ±}.

Simultaneous, Single-Unit Auctions

The summary in table 1 highlights the problem with trying to identify polarization using bids from a single-unit auction due to the confounded information provided by optimal bids. This identification problem may be overcome by using simultaneous, single-unit auctions with free disposal and without product endowments where the auctioned products are different, but share the polarizing attribute (e.g., Huffman et al., 2003).

Suppose an individual participates in $K^S > 1$ incentive-compatible auctions where $K \geq K^S$. Let the auctioned products be denoted by $x^e_k = 1$ for $k = 1, \ldots, K^S$. The optimal bid $b(x^e_k, 0)$ for each $x^e_k$ is defined by equations (5) and (6). Observationally, there are $K^S$ bids, one for each product. In terms of polarization, the only distinguishing characteristic is whether a bid is positive or zero. However, for $K^S$ auctions, there are $2^{K^S}$ possible bid combinations to make inferences about polarization. If $b(x^e_k, 0) > (=) 0$ for all $k = 1, \ldots, K^S$, then $x$ is not negatively (positively) polarized, implying the information set $\{±, +\}(\{−, ±\})$. For the remaining $2^{K^S} - 2$ bid combinations with some positive and some zero bids, $x$ is not polarized, implying the information set $\{±\}$—bids are completely informative for no polarization. This mix of positive and zero bids makes it possible to distinguish an individual without polarization from one with positive or negative polarization at least some of the time, which is the key to identification.

To see how this information can be used to estimate the probability of positive and negative polarization, suppose $N$ individuals participate in simultaneous, single-unit auctions. Equation (5) implies that the $i$th individual’s WTP for $x^e_k$ is

$$w_{ki} = m_i - e_{ki}(x^e_k, p_i, V_i(p_i, m_i)),$$

where the subscript $i$ denotes individual specific preferences, prices, and income. This WTP will never exceed what could be paid for the product “in the field,” which is the field-price censoring described in Harrison, Harstad, and Rutstrom (2004), and will never be less than zero due to free disposal: $p_h z^e_h \geq w_{ki} \geq 0$.\(^7\) Equation (8) can also be written as

$$w_{ki} = \mu_k + \epsilon_{ki},$$

\(^7\) This result is proven in the appendix as Proposition A6.
where $\mu_k$ is the expected WTP for all individuals and $\epsilon_{ki}$ is how an individual’s WTP differs due to differences in preferences, prices, and income. Equation (9) and the bounds on the WTP imply that $p_k x^*_k - \mu_k \geq \epsilon_{ki} > -\mu_k$ for positive polarization, $\epsilon_{ki} = -\mu_k$ for negative polarization, and $p_k x^*_k - \mu_k \geq \epsilon_{ki} \geq -\mu_k$ for no polarization. The implication of this result is that appropriate assumptions for characterizing the distribution of the WTP for econometric analysis depend on polarization.

Let $\mu_k(\tau)$ be the expected WTP, $f_k(\epsilon_{ki}|\tau)$ be the density of $\epsilon_{ki}$, and $F_k(\epsilon_{ki}|\tau)$ be the distribution of $\epsilon_{ki}$ conditional on polarization $\tau \in \{\pm, +\}$. Also, let $q(\tau) \geq 0$ be the probability of polarization $\tau$ such that $q(+) + q(\pm) + q(-) = 1$. Individuals who submit all zero bids are not positively polarized (i.e., the information set is $\{-\} \cup \{\pm\}$). The probability of this is

$$\Pr_i(\{-\}) = q(-) + q(\pm) \prod_{k=1}^{K} F_k(-\mu_k(\pm)|\pm),$$

assuming the independence of errors conditional on polarization. The first term on the right side of equation (10) is the probability of an individual with negative polarization who always submits zero bids. The second term is the probability of an individual with no polarization also submitting all zero bids. Individuals who submit all positive bids are not negatively polarized (i.e., the information set is $\{\pm, +\}$). The probability of this is

$$\Pr_i(\{\pm, +\}) = q(\pm) \prod_{k=1}^{K} f_k(b_i(x^*_k, 0) - \mu_k(\pm)|\pm) + q(+) \prod_{k=1}^{K} f_k(b_i(x^*_k, 0) - \mu_k(+)\mid +),$$

where $b_i(x^*_k, 0)$ is the individual’s bid for the $k$th product. The first term on the right side of equation (11) is the probability of an individual with no polarization multiplied by the probability of the observed positive bids given no polarization. The second term is the probability of an individual with positive polarization multiplied by the probability of the observed positive bids given positive polarization. Individuals who submit a mix of positive and zero bids have no polarization (i.e., the information set is $\{\pm\}$), which occurs with probability

$$\Pr_i(\{\pm\}) = q(\pm) \prod_{k=1}^{K} (d_{ki} f_k(b_i(x^*_k, 0) - \mu_k(\pm)|\pm) + (1 - d_{ki}) F_k(-\mu_k(\pm)|\pm),$$

where $d_{ki} = 1$ if individual $i$ submitted a positive bid for product $k$ and zero otherwise. Equation (12) includes the probability of an individual with no polarization multiplied by the probability of the observed positive and zero bids given no polarization. The log-likelihood function is

$$L = \sum_{\kappa \in \{\{+, \pm\}, \{\pm, -\}, \{\pm\}\}} \sum_{i \in \Omega(\kappa)} \ln(\Pr_i(\kappa)),$$

where $\Omega(\{-\}), \Omega(\{\pm, +\})$, and $\Omega(\{\pm\})$ are the sets of individuals who submitted all zero, all positive, and a mix of positive and zero bids. Equations (10)–(13) describe a finite-mixture econometric model. This log-likelihood function can be optimized to identify the probability of positive and negative polarization and the parameters of the conditional WTP distributions given positive and no polarization.

Before turning to an empirical application, we briefly compare our finite-mixture model to the double-hurdle and Tobit models that have commonly been used to analyze bids in the homegrown value-auction literature. Double-hurdle models have been used with single-unit-auction bids to capture the probability that an individual bids zero along with the distribution of the WTP conditional on a positive bid. The analysis is similar to our finite-mixture model under the assumption that positive polarization does not exist. It is different because the model does not distinguish between zero bids attributable to individuals with negative polarization and zero bids attributable to
individuals with no polarization, so there is no way to separately identify the probability of negative polarization.

Tobit models have also been used with single-unit auctions to account for zero bids. Tobit models assume a normal distribution with unconstrained errors. The implication of unconstrained errors is that there is always some positive probability of observing a positive bid and some positive probability of observing a zero bid. These assumptions are contrary to negative polarization, which implies a zero probability of observing a positive bid, and positive polarization, which implies a zero probability of observing a zero bid. Therefore, Tobit models assume from the outset that polarization does not exist. The random-effects and SUR Tobit models used previously in the literature to analyze bids from simultaneous, single-unit auctions also assume that polarization does not exist from the outset, though these models do permit other types of correlation between subjects’ bids, which are not considered in our finite-mixture model.

**Polarization in an Ornamental Plant Auction**

Yue, Hurley, and Anderson (2011) report the results of a April 2007 homegrown value-auction experiment conducted in St. Paul, Minnesota, with seventy-six individuals recruited based on their interest in gardening. The primary purpose of the experiment was to determine whether labeling plants based on their invasive and noninvasive attributes would discourage the purchase of invasive plants. This hypothesis was supported by survey data in which 98% of the respondents said they would not buy plants labeled as invasive (Reichard and White, 2001).

In the experiment, two-round, simultaneous, second-price, single-unit Vickery auctions were used to elicit the WTP for ten plants—five invasive and five noninvasive. The plants were paired such that plants in a pair were almost identical in appearance, but one was invasive and the other was noninvasive. Different pairs of plants differed notably in size and appearance. To control for field-price censoring (Harrison, Harstad, and Rutstrom, 2004), the selected plants were not available from local retailers and were grown from seed in the University of Minnesota greenhouses. Individuals bid on all ten plants in the first round without being told whether they were invasive or noninvasive, but with informational labels similar to those provided by retailers. They were told that only one auction in the experiment would be binding, which is a strategy commonly used to control demand-reduction bias (Melton et al., 1996; List and Lucking-Reiley, 2000). They bid on all ten plants again in the second round, with the plant label also revealing whether it was invasive or noninvasive. After the first round, individuals were not shown any bid information before bidding in the second round to control for bid-affiliation bias (Harrison, Harstad, and Rutstrom, 2004). At the end of the experiment, individuals received $30 for their participation. If they won an auction, they paid the second highest price and were given the plant to take home.

Figure 2 shows the distribution of individual bids in each round for each of the invasive and noninvasive plants as well as average bids. These results show a prevalence of zero bids that is most remarkable for the invasive plants in round 2, when individuals knew these plants were invasive; more than one out of four bids were zero for each plant. Therefore, negative polarization seems reasonably likely for plants labeled as invasive in round 2.

**Implementation of the Finite-Mixture Model**

We explored the polarization hypothesis by applying the proposed finite-mixture model to the invasive and noninvasive plant bids in rounds 1 and 2. Each attribute and round was analyzed separately, which is most appropriate given the specification of the model. A cursory look at individual bidding behavior revealed that there were individuals with a mix of positive and zero bids for both attributes in both rounds, which means we cannot reject the hypothesis that some individuals exhibited no polarization ($q(\pm) > 0$). Therefore, the null hypothesis of interest was whether $q(+) = q(-) = 0$, implying there was no positive or negative polarization.
The finite-mixture model was implemented assuming $f_k(b_i(x_k^e,0) - \mu_k(\pm))$ was log-normal to constrain the WTP to be greater than zero.\(^8\) For $f_k(b_i(x_k^e,0) - \mu_k(\pm))$ and $F_k(-\mu_k(\pm))$, a truncated normal density and distribution were used to accommodate a zero WTP. The log-likelihood function was programmed and optimized using STATA. Since it is common for finite-mixture models to have local optima (Titterington, Smith, and Makov, 1985), we used a range of starting values in the optimization.

To test $q(+) = q(-) = 0$, we estimated the model for each attribute and round with and without this restriction and computed the likelihood-ratio statistics. Critical values for these statistics were calculated using the parametric bootstrap method described in Schlattmann (2009), since this restriction lies on the boundary of the parameter space, rendering the typical $\chi^2$ critical values inappropriate (Titterington, Smith, and Makov, 1985). This parametric bootstrap method uses the estimates of the unrestricted model to repeatedly simulate individual bids, assuming the model is correct. It then re-estimates the unrestricted and restricted models for each simulated replication and uses the outcomes to construct a distribution for the likelihood ratio and other statistics of interest. This method was also employed to calculate confidence intervals for the WTP, probability of zero bids, and probability of positively and negatively polarized preferences.

**Results**

Table 2 reports estimated probabilities of positive and negative polarization and the likelihood ratio statistic by attribute and round. The likelihood ratio statistics for both attributes and rounds exceeds the 99% critical values, so we can reject the hypothesis that there was no positive or negative polarization. For both attributes and rounds, about one-third of the individuals were estimated to have positive polarization, which is significantly greater than zero based on the 95% confidence intervals. The probability of an individual having negative polarization was 0.013 for the invasive

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\(^8\) A truncated normal was also explored, but it did not fit the data as well.
Table 2. Estimated Probabilities [95% Confidence Interval] of Positive and Negative Polarization and Likelihood Ratio Test that these Probabilities are Jointly Zero

<table>
<thead>
<tr>
<th></th>
<th>Invasive Attribute</th>
<th>Noninvasive Attribute</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Round 1</td>
<td>Round 2</td>
</tr>
<tr>
<td>Positively Polarized</td>
<td>0.387 [0.261, 0.518]</td>
<td>0.384 [0.275, 0.495]</td>
</tr>
<tr>
<td>Negatively Polarized</td>
<td>0.013 [0.000, 0.039]</td>
<td>0.144 [0.065, 0.223]</td>
</tr>
<tr>
<td>Likelihood Ratio Test for No Positively &amp; Negatively Polarized Preferences</td>
<td>190.93</td>
<td>265.80</td>
</tr>
<tr>
<td>99% Critical Value for Likelihood Ratio</td>
<td>49.79</td>
<td>46.01</td>
</tr>
</tbody>
</table>

Notes: Confidence intervals and critical values constructed using parametric bootstrap with 500 replications.

The consistent level of positive polarization across attributes and rounds suggests that positive polarization is related more to an attribute shared by all auctioned products rather than the invasive or noninvasive attribute, which reveals a weakness in the experimental design for quantifying positive polarization for the invasive or noninvasive attribute. We hypothesize that the identified positive polarization relates to the ornamental plant attribute, not the invasive or noninvasive attribute, since individuals were recruited based on their interest in gardening. Unfortunately, this hypothesis cannot be formally tested since all of the auctioned products were ornamental plants. The increase in the probability of negative polarization from round 1 to round 2—after individuals learned these plants possessed the invasive attribute—supports the hypothesis that some individuals have negative polarization when they know an ornamental plant is invasive, although not nearly to the extent reported by Reichard and White (2001).

Table 3 reports estimates of the mean WTP by plant and 95% confidence intervals for individuals with positive and no polarization. It also reports the estimated probability of a zero bid by plant. The mean WTP for all plants is higher for individuals with positive polarization compared to no polarization. For the invasive attribute, the mean WTP for all plants was higher in round 1 compared to round 2, regardless of individuals having positive or no polarization. The probability of a zero bid is higher in round 2 than in round 1. For the noninvasive attribute, opposite results emerge. The mean WTP for all plants is higher in round 2, while the probabilities of a zero bids are modestly lower.

Discussion

An important insight provided by our finite-mixture model is the distinction between the probability of a zero bid for individual invasive plants and the probability of negative polarization for all invasive plants. Why this insight is important can be illustrated with an example. Table 4 provides hypothetical preferences over two plants with invasive labels for two consumers with and without polarization. To make this illustration as stark as possible, assume these consumers are only interested in purchasing one plant at most.

9 Adding nonplant invasive and noninvasive products to the auction would allow this hypothesis to be formally tested.

10 An alternative to the negative polarization hypothesis that might explain our results is the notion of “unengaged” bidders proposed by Lusk and Fox (2003). “Unengaged” bidders bid zero for all auction products in an experiment, which was true for only one out of our seventy-six participants. However, even if we assume that this subject was unengaged rather than negatively polarized, our estimate of the probability of negative polarization is still significantly greater than the maximum possible probability of “unengaged” bidders: 0.013 = 1/76.
Table 3. Estimated Willingness to Pay [95% confidence interval] for Positive and No Polarization, and the Estimated Probability of a Zero Bid [95% confidence interval]

<table>
<thead>
<tr>
<th>Plant</th>
<th>Round 1</th>
<th>Round 2</th>
<th>Round 1</th>
<th>Round 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>WTP Given Positive Polarization ($)</td>
<td></td>
<td>WTP Given No Polarization ($)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Invasive</td>
<td>Noninvasive</td>
<td>Invasive</td>
<td>Noninvasive</td>
</tr>
<tr>
<td>Plant 1</td>
<td>2.35</td>
<td>1.66</td>
<td>2.11</td>
<td>2.64</td>
</tr>
<tr>
<td></td>
<td>[1.71, 3.30]</td>
<td>[1.10, 2.59]</td>
<td>[1.46, 3.04]</td>
<td>[1.77, 4.01]</td>
</tr>
<tr>
<td>Plant 2</td>
<td>2.87</td>
<td>2.22</td>
<td>3.01</td>
<td>3.47</td>
</tr>
<tr>
<td></td>
<td>[2.47, 3.27]</td>
<td>[1.63, 3.09]</td>
<td>[2.37, 3.67]</td>
<td>[2.64, 4.63]</td>
</tr>
<tr>
<td>Plant 3</td>
<td>4.05</td>
<td>3.50</td>
<td>3.64</td>
<td>3.92</td>
</tr>
<tr>
<td></td>
<td>[3.44, 4.66]</td>
<td>[2.61, 4.76]</td>
<td>[2.99, 4.40]</td>
<td>[3.11, 4.88]</td>
</tr>
<tr>
<td>Plant 4</td>
<td>5.37</td>
<td>3.90</td>
<td>5.57</td>
<td>6.42</td>
</tr>
<tr>
<td></td>
<td>[4.50, 6.64]</td>
<td>[3.18, 4.85]</td>
<td>[4.47, 7.00]</td>
<td>[5.04, 8.16]</td>
</tr>
<tr>
<td>Plant 5</td>
<td>4.62</td>
<td>3.76</td>
<td>4.76</td>
<td>5.95</td>
</tr>
<tr>
<td></td>
<td>[3.81, 5.43]</td>
<td>[3.09, 4.65]</td>
<td>[3.86, 5.69]</td>
<td>[4.42, 7.89]</td>
</tr>
<tr>
<td>Plant 1</td>
<td>0.64</td>
<td>0.32</td>
<td>0.39</td>
<td>0.55</td>
</tr>
<tr>
<td></td>
<td>[0.43, 0.86]</td>
<td>[0.20, 0.45]</td>
<td>[0.26, 0.53]</td>
<td>[0.39, 0.71]</td>
</tr>
<tr>
<td>Plant 2</td>
<td>0.66</td>
<td>0.36</td>
<td>0.55</td>
<td>0.81</td>
</tr>
<tr>
<td></td>
<td>[0.44, 0.89]</td>
<td>[0.21, 0.53]</td>
<td>[0.38, 0.75]</td>
<td>[0.59, 1.06]</td>
</tr>
<tr>
<td>Plant 3</td>
<td>1.01</td>
<td>0.52</td>
<td>0.91</td>
<td>1.08</td>
</tr>
<tr>
<td></td>
<td>[0.66, 1.36]</td>
<td>[0.30, 0.80]</td>
<td>[0.60, 1.29]</td>
<td>[0.77, 1.44]</td>
</tr>
<tr>
<td>Plant 4</td>
<td>0.95</td>
<td>0.29</td>
<td>1.73</td>
<td>2.10</td>
</tr>
<tr>
<td></td>
<td>[0.62, 1.27]</td>
<td>[0.18, 0.42]</td>
<td>[1.18, 2.23]</td>
<td>[1.59, 2.63]</td>
</tr>
<tr>
<td>Plant 5</td>
<td>1.36</td>
<td>0.36</td>
<td>1.69</td>
<td>1.95</td>
</tr>
<tr>
<td></td>
<td>[0.96, 1.79]</td>
<td>[0.21, 0.52]</td>
<td>[1.21, 2.17]</td>
<td>[1.47, 2.49]</td>
</tr>
</tbody>
</table>

Notes: Confidence intervals constructed using parametric bootstrap with 500 replications.

Without polarization, Consumer A is willing to pay up to $5 for the first plant and nothing for the second. Alternatively, Consumer B is willing to pay nothing for the first plant and up to $6 for the second. Therefore, the average WTP is $2.50 for the first and $3 for the second plant. The probability of a zero bid in a single-unit auction is 0.5 for both plants, while the probability of negative and positive polarization is 0.0. Now suppose the price for both plants is $4. Consumer A will purchase the first plant and Consumer B the second for a total sale of two invasive plants.

With polarization, Consumer A is willing to pay up to $5 for the first and up to $6 for the second plant. Consumer B is willing to pay nothing for either plant. The average WTP is the same as before, $2.50 for the first and $3 for the second plant. The probability of a zero bid in a single-unit auction is also the same as before, 0.5 for both plants. What is different from before is the probability of
negative and positive polarization, which is now greater than zero for both. The number of plants sold at a price of $4 is also different. Consumer A will purchase the second plant and Consumer B will not make a purchase for a total sale of one instead of two invasive plants.

Ignoring polarization and looking at the WTP distribution for each plant separately, these examples are exactly the same, so there is no way to distinguish between the very different market outcomes that could result. Taking polarization into account and looking at the joint distribution of the WTPs, it becomes possible to distinguish between these different market outcomes. While the random-effects and SUR Tobit models used previously in the literature provide estimates of the joint distribution of the WTPs, they are not able to quantify the extent of polarization because they effectively assume that polarization does not exist.

**Conclusions**

Homegrown value auctions provide a useful tool for understanding consumer preferences for a particular product attribute and how these preferences might change based on what consumers know about the attribute. The difference in WTP for a product with and without the attribute of interest and the difference in WTP for a product with different information on a particular attribute have been of particular interest. Less attention has been devoted to understanding the willingness of consumers to purchase any products with the attribute of interest, which is also of potential interest to researchers, marketers, and policy makers.

This article formally introduces the concept of preference polarization and demonstrates how it can be used to gain additional insight into the implications of zero bids in homegrown value auctions. Preference polarization focuses on the desirability of an attribute shared by multiple products rather than on the desirability of a specific product with the attribute. Preferences are negatively polarized if all products with the attribute are strictly undesirable and positively polarized if all products with the attribute are strictly desirable.

After formally characterizing polarization and bidding behavior in an incentive-compatible auction, we show the implications on bidding behavior of auction designs where one product is traded for another. We then show how single-unit auctions provide little information for quantifying polarization, while simultaneous, single-unit auctions without an endowment and with free disposal
make the quantification of polarization possible with a finite-mixture econometric model. Finally, the finite-mixture model was applied to data from a homegrown value auction conducted with five invasive and five noninvasive ornamental plants in order to estimate the extent of polarization for the invasive and noninvasive attributes with and without the labeling of these attributes. The results of this analysis show how the quantification of polarization can provide additional information about consumer demand. For example, while about one in four individuals were unwilling to purchase a particular plant that was labeled as invasive, only about one in seven were unwilling to purchase any plants labeled as invasive. Individuals who were willing to buy all of the invasive plants were willing to pay more on average for any particular plant than individuals who were willing to buy some invasive plants, but not others. Such information makes it possible to more accurately assess how labeling invasive plants could affect their demand.

Our theory provides a novel and rigorous framework for future explorations of the implications of alternative experimental designs on bidding behavior in incentive-compatible auctions. Our proposed finite-mixture model makes it possible to quantify the extent of polarization when combined with simultaneous, single-unit auction designs. Extensions of our finite-mixture model for future work include testing for changes in polarization due to labeling and further accounting for potential correlation in bids across auctions conditional on positive and no polarization. The greatest challenges to achieving these and other useful extensions of our finite-mixture model are likely to be computational rather than conceptual.

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References


Appendix

The first two propositions formally state the properties of the indirect utility function with respect to income $m$ and the endowment $z_h$ for use in the proofs of Propositions A3–A6. The proofs of these properties closely mirror the textbook proofs of the properties of indirect utility functions and are available from the authors upon request.

**Proposition A1:** With free disposal,

(a) $V_h(z_h', p, m') > V_h(z_h', p, m)$ for all $m' > m$,

(b) $V_h(z_h', p, m) = V_h(0, p, m)$ for all $z_h' > 0$ if $z_h$ is strictly undesirable, and

(c) $V_h(z_h', p, m) > V_h(z_h', p, m)$ for all $z_h' > z_h'$ if $z_h$ is strictly desirable.

**Proposition A2:** Without free disposal,

(a) $V_h(z_h', p, m') > V_h(z_h', p, m)$ for all $m' > m$ if $z_h$ is locally nonsatiated,

(b) $V_h(z_h', p, m) > V_h(z_h', p, m)$ for all $z_h' > z_h'$ if $z_h$ is strictly undesirable, and

(c) $V_h(z_h', p, m) > V_h(z_h', p, m)$ for all $z_h' > z_h'$ if $z_h$ is strictly desirable.

**Proposition A3:** In an incentive-compatible auction without a product endowment, the optimal bid is positive (zero) for a strictly desirable (undesirable) product.

**Proof:** Suppose the product $z_h > 0$ is strictly desirable such that $V_h(z_h', p, m - w) > V_h(0, p, m)$ for $w = 0$ by Propositions A1 and A2 (c). Propositions A1 and A2 (a) and equations (5) and (6) then imply that $b(z_h', z_h') > 0$. Suppose the product $z_h > 0$ is strictly undesirable. With free disposal, $V_h(z_h', p, m - w) = V_h(z_h', p, m)$ for $w = 0$ by Proposition A1 (b) such that $V_h(z_h', p, m - w) < V_h(0, p, m)$ for $w > 0$ by Proposition A1 (a). Therefore, equations (5) and (6) imply that $b(z_h', z_h') = 0$. Without free disposal, $V_h(z_h', p, m) < V_h(0, p, m)$ by Proposition A2 (b) and $V_h(z_h', p, m - w) < V_h(z_h', p, m)$ for $w > 0$ by Proposition A2 (a). Therefore, $V_h(z_h', p, m - w) < V_h(0, p, m)$ for $w > 0$ such that equations (5) and (6) imply that $b(z_h', z_h') = 0$. Q.E.D.

**Proposition A4:** In an incentive-compatible auction with an endowment and free disposal, the optimal bid (a) is positive if the auctioned product is strictly desirable and the endowed product is strictly undesirable and (b) zero if the auctioned product is strictly undesirable.

**Proof:** First consider Proposition A4 (a). Proposition A1 (b) implies that $V_h(z_h', p, m) = V_h(0, p, m)$, while Proposition A1 (c) implies that $V_h(z_h', p, m - w) > V_h(0, p, m) = V_h(0, p, m)$ for $w = 0$. Proposition A1 (a) and equations (5) and (6) then imply that $b(z_h', z_h') > 0$. For Proposition A4 (b), Proposition A1 (b) implies that $V_h(z_h', p, m - w) = V_h(0, p, m - w)$. Proposition A1 (a) implies that $V_h(0, p, m - w) < V_h(0, p, m) = V_h(0, p, m)$ for $w > 0$ and $V_h(0, p, m - w) = V_h(0, p, m) - w$ for $w = 0$, but equations (5) and (6) then imply that $b(z_h', z_h') = 0$. Q.E.D.

**Proposition A5:** In an incentive-compatible auction with an endowment and without free disposal, the optimal bid (a) is positive if the auctioned product is strictly desirable and the endowed product is strictly undesirable and (b) zero if the auctioned product is strictly undesirable and the endowed product is strictly desirable.

**Proof:** For Proposition A5 (a), Proposition A2 (b) implies that $V_h(0, p, m) > V_h(z_h', p, m)$, while Proposition A2 (c) implies that $V_h(z_h', p, m - w) > V_h(0, p, m) = V_h(0, p, m)$ for $w = 0$. Proposition A2 (a) and equations (5) and (6) then imply that $b(z_h', z_h') > 0$. For Proposition A5 (b), Proposition A2 implies that $V_h(z_h', p, m) > V_h(0, p, m) = V_h(0, p, m) > V_h(0, p, m - w) = V_h(z_h', p, m - w)$ for $w > 0$, but equations (5) and (6) then imply that $b(z_h', z_h') = 0$. Q.E.D.

---

11 For $z \in \mathbb{R}^{K+L}$, $z_h \in \mathbb{R}^{K+L-1}$ is locally nonsatiated if for every $z_h \in \mathbb{R}^+$, $z_{-h} \in \mathbb{R}^{K+L-1}$ and $\epsilon > 0$ there is $z_{-h} \in \mathbb{R}^{K+L-1}$ such that $||z_{-h} - z_{-h}|| \leq \epsilon$ and $(z_h, z_{-h}) \succ (z_h, z_{-h})$ where $|| \cdot ||$ represents the Euclidian norm and $\succ$ represents strict preference.
Proposition A6: An individual’s maximum willingness to pay, $w_{ki}$, in an auction with free disposal and no endowment will satisfy $p_h z_h^e \geq w_{ki} \geq 0$.

Proof: Suppose this is not the case, such that $w_{ki} < 0$ or $w_{ki} > p_h z_h^e$. First consider $w_{ki} < 0$. By equation (5) and the assumption of no product endowment, $w_{ki} = \max \{ w \in R : V_h(z_h^e, p, m - w) \geq V_h(0, p, m) \}$. Propositions A1 (b) and (c) imply that $V_h(z_h^e, p, m) \geq V_h(0, p, m)$ such that $0 \in \{ w \in R : V_h(z_h^e, p, m - w) \geq V_h(0, p, m) \}$, implying the contradiction $w_{ki} \geq 0$. Proposition A1 (a) and $w_{ki} > p_h z_h^e$ imply that $V_h(z_h^e, p, m - p_h z_h^e) > V_h(z_h^e, p, m)$. By equations (3) and (4), $V_h(z_h^e, p, m - p_h z_h^e) > V_h(z_h^e, p, m - w)$ implies that $U(z') > U(z'')$ for all $z' \in z_h(z_h^e, p, m - p_h z_h^e)$ and $z'' \in z_h(z_h^e, p, m - w)$. Also by equations (3) and (4), $V_h(z_h^e, p, m - w) \geq V_h(0, p, m)$ for $w \geq 0$ such that $U(z') \geq U(z'')$ for all $z'' \in z_h(0, p, m)$. Equation (1) and $z \in z_h(z_h^e, p, m - p_h z_h^e)$ imply that $z \in B_h(0, p, m)$. However, $z \in B_h(0, p, m)$, $z'' \in z_h(0, p, m)$, and equation (3) then imply that $U(z'') \geq U(z)$, which yields the contradiction $U(z') \geq U(z'') \geq U(z)$. Q.E.D.